

Nonminimal De Rham-Hodge Operators and Non-commutative Residue

Abstract

In this paper, we get a Kastler-Kalau-Walze type theorem associated to nonminimal de Rham-Hodge operators on compact manifolds with boundary. We give two kinds of operator-theoretic explanations of the gravitational action in the case of four dimensional compact manifolds with flat boundary.

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1. Introduction

The noncommutative residue found in [1, 2] plays a prominent role in noncommutative geometry. For one dimensional manifolds, the noncommutative residue was discovered by Adler [3] in connection with geometric aspects of nonlinear partial differential equations. In [4], Connes used the noncommutative residue to derive a conformal four dimensional Polyakov action analogy. Moreover, in [5], Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which we call the Kastler-Kalau-Walze theorem. In [6], Kastler gave a brute-force proof of this theorem. Then Kalau and Walze also gave a proof of this theorem by using normal coordinates [7].

In [8], Fedosov etc. defined a noncommutative residue on Boutet de Monvel's algebra and proved that it was a unique continuous trace. In [9], Schrohe gave the relation between the Dixmier trace and the noncommutative residue for manifolds with boundary. For an oriented spin manifold M with boundary ∂M , by the composition formula in Boutet de Monvel's algebra and the definition of $\widetilde{\text{Wres}}$ [10], $\widetilde{\text{Wres}}[(\pi^+ D^{-1})^2]$ should be the sum of two terms from interior and boundary of M , where $\pi^+ D^{-1}$ is an element in Boutet de Monvel's algebra [10]. For lower-dimension spin manifolds with boundary and the associated Dirac operators, Wang computed the lower dimensional volume and got a Kastler-Kalau-Walze type theorem in [11], [12] and [13]. In [14], Gilkey, Branson and Fulling obtained a formula about heat kernel expansion coefficients of nonminimal operators. In [15], we considered the non-commutative residue of nonminimal operators and got the Kastler-Kalau-Walze type theorems for nonminimal operators. The motivation of this paper is to generalize Theorem 2.2 in [15] to manifolds with boundary in the four dimensional case. For this purpose, we introduce the nonminimal de Rham-Hodge operators $\tilde{D} = ad + b\delta$ and nonminimal laplacian operators $\tilde{D}\tilde{D}^* = a^2d\delta + b^2\delta d$. Our main result is as follows:

Theorem 3.5 The following equality holds:

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}^{-1} \circ \pi^+ (\tilde{D}^*)^{-1}] = 4\pi \int_M \sum_{k=0}^4 c_1(4, k, a, b) R d\text{vol}(M) - \frac{23}{12} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \pi \int_{\partial M} K \Omega_3 dx',$$

(for related definition, see Section 3).

This paper is organized as follows: In Section 2, we define lower dimensional volumes of manifolds with boundary for nonminimal De-Rham Hodge operators. In Section 3, for four dimensional compact manifolds with boundary and the associated nonminimal De-Rham Hodge operators $\tilde{D} = ad + b\delta$ and $\tilde{D}^* = bd + a\delta$, we compute the lower dimensional volume $\text{Vol}_4^{(1,1)}$ and get a Kastler-Kalau-Walze type theorem in this case. In Section 4, for four dimensional compact manifolds with boundary and the associated nonminimal De-Rham Hodge operators $\tilde{D} = ad + b\delta$, we compute the lower dimensional volume. When ∂M is flat, we give two kinds of operator theoretic explanations of the gravitational action on boundary.

2. Lower dimensional volumes of Riemannian manifolds with boundary

In this section we consider an n -dimensional oriented Riemannian manifold (M, g^M) equipped with some spin structure. Let M be an n -dimensional compact oriented manifold with boundary ∂M . We assume that the metric g^M on M has the following form near the boundary

$$g^M = \frac{1}{h(x_n)} g^{\partial M} + dx_n^2,$$

where $g^{\partial M}$ is the metric on ∂M . Let $U \subset M$ be a collar neighborhood of ∂M which is diffeomorphic $\partial M \times [0, 1)$. By the definition of $h(x_n) \in C^\infty([0, 1))$ and $h(x_n) > 0$, there exists $\tilde{h} \in C^\infty((-\varepsilon, 1))$ such that $\tilde{h}|_{[0, 1)} = h$ and $\tilde{h} > 0$ for some sufficiently small $\varepsilon > 0$. Then there exists a metric \hat{g} on $\hat{M} = M \cup_{\partial M} \partial M \times (-\varepsilon, 0]$ which has the form on $U \cup_{\partial M} \partial M \times (-\varepsilon, 0]$

$$\hat{g} = \frac{1}{\tilde{h}(x_n)} g^{\partial M} + dx_n^2,$$

such that $\hat{g}|_M = g$. We fix a metric \hat{g} on the \hat{M} such that $\hat{g}|_M = g$.

Let ∇ denote the Levi-civita connection about g^M . In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{e}_1, \dots, \tilde{e}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla(\tilde{e}_1, \dots, \tilde{e}_n) = (\tilde{e}_1, \dots, \tilde{e}_n)(\omega_{s,t}).$$

Let $c(e_i) = \epsilon(e_i) - \iota(e_i)$, $\hat{c}(\tilde{e}_i) = \epsilon(e_i) + \iota(e_i)$. Denote the exterior and interior multiplications by $\epsilon(e_j^*)$, $\iota(e_j^*)$ respectively, denote by $d + \delta : \wedge^*(T^*M) \rightarrow \wedge^*(T^*M)$ the signature operator. By [17], we have

$$d + \delta = \sum_{i=1}^n c(e_i) \left[e_i + \frac{1}{4} \sum_{s,t} \omega_{s,t}(e_i) [\hat{c}(e_s) \hat{c}(e_t) - c(e_s) c(e_t)] \right].$$

Then we define the nonminimal de Rham-Hodge operators as

$$\tilde{D} = ad + b\delta, \quad \tilde{D}^* = bd + a\delta,$$

where \tilde{D}^* is the adjoint operator of \tilde{D} and $ab \neq 0$.

To define the lower dimensional volume, some basic facts and formulae about Boutet de Monvel's calculus which can be found in Sec.2 in [10] and [16] are needed. Let

$$F : L^2(\mathbf{R}_t) \rightarrow L^2(\mathbf{R}_v); \quad F(u)(v) = \int e^{-ivt} u(t) dt$$

denote the Fourier transformation and $\Phi(\overline{\mathbf{R}^+}) = r^+ \Phi(\mathbf{R})$ (similarly define $\Phi(\overline{\mathbf{R}^-})$), where $\Phi(\mathbf{R})$ denotes the Schwartz space and

$$r^+ : C^\infty(\mathbf{R}) \rightarrow C^\infty(\overline{\mathbf{R}^+}); \quad f \rightarrow f|_{\overline{\mathbf{R}^+}}; \quad \overline{\mathbf{R}^+} = \{x \geq 0; x \in \mathbf{R}\}.$$

We define $H^+ = F(\Phi(\overline{\mathbf{R}^+}))$; $H_0^- = F(\Phi(\overline{\mathbf{R}^-}))$ which are orthogonal to each other. We have the following property: $h \in H^+$ (H_0^-) iff $h \in C^\infty(\mathbf{R})$, which has an analytic extension to the lower (upper) complex half-plane $\{\text{Im} \xi < 0\}$ ($\{\text{Im} \xi > 0\}$) such that for all nonnegative integer l ,

$$\frac{d^l h}{d\xi^l}(\xi) \sim \sum_{k=1}^{\infty} \frac{d^l}{d\xi^l} \left(\frac{c_k}{\xi^k} \right)$$

as $|\xi| \rightarrow +\infty, \text{Im} \xi \leq 0$ ($\text{Im} \xi \geq 0$).

Let H' be the space of all polynomials and $H^- = H_0^- \oplus H'$; $H = H^+ \oplus H^-$. Denote by π^+ (π^-) respectively the projection on H^+ (H^-). For calculations, we take $H = \tilde{H} = \{\text{rational functions having no poles on the real axis}\}$ (\tilde{H} is a dense set in the topology of H). Then on \tilde{H} ,

$$\pi^+ h(\xi_0) = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{h(\xi)}{\xi_0 + iu - \xi} d\xi, \quad (2.1)$$

where Γ^+ is a Jordan close curve included $\text{Im}\xi > 0$ surrounding all the singularities of h in the upper half-plane and $\xi_0 \in \mathbf{R}$. Similarly, define π' on \tilde{H} ,

$$\pi' h = \frac{1}{2\pi} \int_{\Gamma^+} h(\xi) d\xi.$$

So, $\pi'(H^-) = 0$. For $h \in H \cap L^1(\mathbf{R})$, $\pi' h = \frac{1}{2\pi} \int_{\mathbf{R}} h(v) dv$ and for $h \in H^+ \cap L^1(\mathbf{R})$, $\pi' h = 0$.

Denote by \mathcal{B} Boutet de Monvel's algebra, we recall the main theorem in [8].

Theorem 2.1. (Fedosov-Golse-Leichtnam-Schrohe) *Let X and ∂X be connected, $\dim X = n \geq 3$, $A = \begin{pmatrix} \pi^+ P + G & K \\ T & S \end{pmatrix} \in \mathcal{B}$, and denote by p , b and s the local symbols of P , G and S respectively. Define:*

$$\begin{aligned} \widetilde{\text{Wres}}(A) &= \int_X \int_{\mathbf{S}} \text{tr}_E [p_{-n}(x, \xi)] \sigma(\xi) dx \\ &\quad + 2\pi \int_{\partial X} \int_{\mathbf{S}'} \{ \text{tr}_E [(\text{tr} b_{-n})(x', \xi')] + \text{tr}_F [s_{1-n}(x', \xi')] \} \sigma(\xi') dx'. \end{aligned} \quad (2.2)$$

Then a) $\widetilde{\text{Wres}}([A, B]) = 0$, for any $A, B \in \mathcal{B}$; b) It is a unique continuous trace on $\mathcal{B}/\mathcal{B}^{-\infty}$.

Let p_1, p_2 be nonnegative integers and $p_1 + p_2 \leq n$. Then by Sec 2.1 of [12], we have

Definition 2.2. *Lower-dimensional volumes of Riemannian manifolds with boundary are defined by*

$$\text{Vol}_n^{(p_1, p_2)} M := \widetilde{\text{Wres}}[\pi^+ \tilde{D}^{-p_1} \circ \pi^+ (\tilde{D}^*)^{-p_2}]. \quad (2.3)$$

Denote by $\sigma_l(A)$ the l -order symbol of an operator A . For n dimensional Riemannian manifolds with boundary, an application of (2.1.4) in [10] shows that

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}^{-p_1} \circ \pi^+ (\tilde{D}^*)^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{\Lambda^*(T^*M)} [\sigma_{-n}(\tilde{D}^{-p_1} (\tilde{D}^*)^{-p_2})] \sigma(\xi) d\xi + \int_{\partial M} \Phi, \quad (2.4)$$

where

$$\begin{aligned} \Phi &= \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \text{trace}_{\Lambda^*(T^*M)} \left[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+ (\tilde{D}^{-p_1})(x', 0, \xi', \xi_n) \right. \\ &\quad \left. \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l((\tilde{D}^*)^{-p_2})(x', 0, \xi', \xi_n) \right] d\xi_n \sigma(\xi') d\xi', \end{aligned} \quad (2.5)$$

and the sum is taken over $r - k + |\alpha| + \ell - j - 1 = -n, r \leq -p_1, \ell \leq -p_2$.

3. A Kastler-Kalau-Walze type theorem of nonminimal de Rham-Hodge operators \tilde{D} and \tilde{D}^*

In this section, we compute the lower dimension volume for four dimension compact connected manifolds with boundary associated to nonminimal de Rham-Hodge operators \tilde{D} and \tilde{D}^* and get a Kastler-Kalau-Walze type formula in this case. Let M be an four dimensional compact oriented connected manifold with boundary ∂M , and the metric g^M on M as above. Note that

$$\tilde{D}\tilde{D}^* = a^2 d\delta + b^2 \delta d$$

is a nonminimal operator on $C^\infty(\Lambda^*(T^*M))$, then $[\sigma_{-4}((\tilde{D}\tilde{D}^*)^{-1})]_M$ has the same expression with the case of without boundary in [15], so locally we can use Theorem 2.2 in [15] to compute the first term. Therefore

$$\int_M \int_{|\xi|=1} \text{trace}_{\Lambda^*(T^*M)} [\sigma_{-4}((\tilde{D}\tilde{D}^*)^{-1})] \sigma(\xi) dx = 4\pi \int_M \sum_{k=0}^4 c_1(4, k, a, b) R \text{dvol}(M), \quad (3.1)$$

where R is the scalar curvature and $c_1(4, k, a, b) = b^{-2} \{ \frac{1}{6} \binom{4}{k} - \binom{2}{k-1} \} + (b^{-2} - a^{-2}) \sum_{j < k} (-1)^{j-k} \{ \frac{1}{6} \binom{4}{j} - \binom{2}{j-1} \}$.

Hence we only need to compute $\int_{\partial M} \Phi$. Firstly, we compute the symbol $\sigma(\tilde{D}^{-1})$ and $\sigma((\tilde{D}^*)^{-1})$. Denote by $\tilde{c}(\xi) = a\epsilon(\xi) - b\iota(\xi)$, $\bar{c}(\xi) = b\epsilon(\xi) - a\iota(\xi)$, then we have $\tilde{c}(e_j^*) = a\epsilon(e_j^*) - b\iota(e_j^*)$ and $\bar{c}(e_j^*) = b\epsilon(e_j^*) - a\iota(e_j^*)$. From the form of Signature operator

$$\tilde{D} = \sum_{i=1}^n \tilde{c}(e_i) \left[e_i + \sum_{s,t} \omega_{s,t}(e_i) (\hat{c}(e_s)\hat{c}(e_t) - c(e_s)c(e_t)) \right], \quad (3.2)$$

we get

$$\sigma_1(\tilde{D}) = \sqrt{-1}\tilde{c}(\xi); \quad (3.3)$$

$$\sigma_0(\tilde{D}) = \frac{1}{4} \sum_{i,s,t} \omega_{s,t}(e_i) \tilde{c}(e_i) [\hat{c}(e_s)\hat{c}(e_t) - c(e_s)c(e_t)], \quad (3.4)$$

where $\xi = \sum_{i=1}^n \xi_i dx_i$ denotes the cotangent vector. Write

$$\tilde{D}_x^\alpha = (-\sqrt{-1})^{|\alpha|} \partial_x^\alpha; \quad \sigma(\tilde{D}) = p_1 + p_0; \quad \sigma(\tilde{D}^{-1}) = \sum_{j=1}^{\infty} q_{-j}. \quad (3.5)$$

By the composition formula of pseudodifferential operators, we have

$$\begin{aligned} 1 = \sigma(\tilde{D} \circ \tilde{D}^{-1}) &= \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha [\sigma(\tilde{D})] D_x^\alpha [\sigma(\tilde{D}^{-1})] \\ &= (p_1 + p_0)(q_{-1} + q_{-2} + q_{-3} + \cdots) \\ &\quad + \sum_j (\partial_{\xi_j} p_1 + \partial_{\xi_j} p_0) (D_{x_j} q_{-1} + D_{x_j} q_{-2} + D_{x_j} q_{-3} + \cdots) \\ &= p_1 q_{-1} + (p_1 q_{-2} + p_0 q_{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} q_{-1}) + \cdots \end{aligned}$$

Thus, we obtain

$$q_{-1} = p_1^{-1}; \quad q_{-2} = -p_1^{-1} [p_0 p_1^{-1} + \sum_j \partial_{\xi_j} p_1 D_{x_j} (p_1^{-1})]. \quad (3.6)$$

By (3.2), (3.5) and direct computations, we have

Lemma 3.1. *Let \tilde{D} , \tilde{D}^* on $C^\infty(\Lambda^*(T^*M))$. Then*

$$q_{-1}(\tilde{D}^{-1}) = \frac{\sqrt{-1}\tilde{c}(\xi)}{ab|\xi|^2}; \quad (3.7)$$

$$q_{-2}(\tilde{D}^{-1}) = \frac{\tilde{c}(\xi)\sigma_0(\tilde{D})\tilde{c}(\xi)}{a^2b^2|\xi|^4} + \frac{\tilde{c}(\xi)}{a^2b^2|\xi|^6} \sum_j \tilde{c}(dx_j) \left[\partial_{x_j} [\tilde{c}(\xi)] |\xi|^2 - \tilde{c}(\xi) \partial_{x_j} (|\xi|^2) \right]; \quad (3.8)$$

$$q_{-1}((\tilde{D}^*)^{-1}) = \frac{\sqrt{-1}\bar{c}(\xi)}{ab|\xi|^2}; \quad (3.9)$$

$$q_{-2}((\tilde{D}^*)^{-1}) = \frac{\bar{c}(\xi)\sigma_0(\tilde{D}^*)\bar{c}(\xi)}{a^2b^2|\xi|^4} + \frac{\bar{c}(\xi)}{a^2b^2|\xi|^6} \sum_j \bar{c}(dx_j) \left[\partial_{x_j} [\bar{c}(\xi)] |\xi|^2 - \bar{c}(\xi) \partial_{x_j} (|\xi|^2) \right], \quad (3.10)$$

where

$$\tilde{p}_0 = \sigma_0(\tilde{D})(x_0) = -\frac{1}{4}h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i)\hat{c}(e_i)\hat{c}(e_n)(x_0) + \frac{1}{4}h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i)c(e_i)c(e_n)(x_0); \quad (3.11)$$

$$\bar{p}_0 = \sigma_0(\tilde{D}^*)(x_0) = -\frac{1}{4}h'(0) \sum_{i=1}^{n-1} \bar{c}(e_i)\hat{c}(e_i)\hat{c}(e_n)(x_0) + \frac{1}{4}h'(0) \sum_{i=1}^{n-1} \bar{c}(e_i)c(e_i)c(e_n)(x_0). \quad (3.12)$$

Since Φ is a global form on ∂M , so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates U of x_0 in ∂M (not in M) and compute $\Phi(x_0)$ in the coordinates $\tilde{U} = U \times [0, 1)$ and the metric $\frac{1}{h(x_n)}g^{\partial M} + dx_n^2$. The dual metric of g^M on \tilde{U} is $h(x_n)g^{\partial M} + dx_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $g_M^{ij} = g^M(dx_i, dx_j)$, then

$$[g_{i,j}^M] = \begin{bmatrix} \frac{1}{h(x_n)}[g_{i,j}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix}; \quad [g_M^{i,j}] = \begin{bmatrix} h(x_n)[g_{i,j}^{\partial M}] & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$\partial_{x_s} g_{ij}^{\partial M}(x_0) = 0, \quad 1 \leq i, j \leq n-1; \quad g_{i,j}^M(x_0) = \delta_{ij}. \quad (3.13)$$

By Lemma 2.2 in [12] and the normal coordinates U of x_0 in ∂M (not in M), we have

Lemma 3.2. *With the metric g^M on M near the boundary*

$$\partial_{x_j}(|\xi|_{g^M}^2)(x_0) = \begin{cases} 0, & \text{if } j < n; \\ h'(0)|\xi'|_{g^{\partial M}}^2, & \text{if } j = n, \end{cases} \quad (3.14)$$

$$\partial_{x_j}[\tilde{c}(\xi)](x_0) = \begin{cases} 0, & \text{if } j < n; \\ \partial_{x_n}(\tilde{c}(\xi'))(x_0), & \text{if } j = n, \end{cases} \quad (3.15)$$

where $\xi = \xi' + \xi_n dx_n$.

Lemma 3.3. [12] *When $i < n$, $\omega_{n,i}(\tilde{e}_i)(x_0) = \frac{1}{2}h'(0)$; and $\omega_{i,n}(\tilde{e}_i)(x_0) = -\frac{1}{2}h'(0)$, In other cases, $\omega_{s,t}(\tilde{e}_i)(x_0) = 0$.*

Lemma 3.4. *By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:*

$$\begin{aligned} \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi')\epsilon(dx_n)](x_0)|_{|\xi'|=1} &= 6ab^2h'(0); \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi')\iota(dx_n)](x_0)|_{|\xi'|=1} = -6a^2bh'(0); \\ \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n)\epsilon(dx_n)](x_0)|_{|\xi'|=1} &= -6b^2ah'(0); \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n)\iota(dx_n)](x_0)|_{|\xi'|=1} = 6a^2bh'(0); \\ \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi')\epsilon(\xi')](x_0)|_{|\xi'|=1} &= -2b^2ah'(0); \text{tr}[\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi')\iota(\xi')](x_0)|_{|\xi'|=1} = 10a^2bh'(0); \\ \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n)\epsilon(\xi')](x_0)|_{|\xi'|=1} &= -10b^2ah'(0); \text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n)\iota(\xi')](x_0)|_{|\xi'|=1} = 2a^2bh'(0), \end{aligned} \quad (3.16)$$

others vanishes.

Proof. By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then

$$\begin{aligned} &\text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi')\epsilon(dx_n)](x_0)|_{|\xi'|=1} \\ &= \text{tr}[(a\epsilon(\xi') - b\iota(\xi'))\tilde{p}_0(a\epsilon(\xi') - b\iota(\xi'))\epsilon(dx_n)](x_0)|_{|\xi'|=1} \\ &= ab\text{tr}[\epsilon(\xi')\iota(\xi')\tilde{p}_0\epsilon(dx_n)](x_0)|_{|\xi'|=1} + ab\text{tr}[\iota(\xi')\epsilon(\xi')\tilde{p}_0\epsilon(dx_n)](x_0)|_{|\xi'|=1} \\ &= ab\text{tr}[\tilde{p}_0\epsilon(dx_n)](x_0)|_{|\xi'|=1}, \end{aligned} \quad (3.17)$$

where

$$\begin{aligned} &\text{tr}[\tilde{p}_0\epsilon(dx_n)] \\ &= \text{tr}\left[\left(-\frac{1}{4}h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i)\hat{c}(e_i)\hat{c}(e_n) + \frac{1}{4}h'(0) \sum_{i=1}^{n-1} \tilde{c}(e_i)c(e_i)c(e_n)\right)\epsilon(dx_n)\right] \\ &= -\frac{1}{4}h'(0) \sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)\tilde{c}(e_i)\hat{c}(e_i)\hat{c}(e_n)] + \frac{1}{4}h'(0) \sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)\tilde{c}(e_i)c(e_i)c(e_n)]. \end{aligned} \quad (3.18)$$

By the relation of the Clifford action and $\epsilon(e_i)\iota(e_j) + \iota(e_j)\epsilon(e_i) = \delta_{ij}$, we obtain

$$\begin{aligned} \sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)\tilde{c}(e_i)\hat{c}(e_i)\hat{c}(e_n)] &= a \sum_{i=1}^{n-1} \text{tr}[\epsilon(e_i)\iota(e_i)\iota(e_n)\epsilon(e_n)] - b \sum_{i=1}^{n-1} \text{tr}[\iota(e_i)\epsilon(e_i)\iota(e_n)\epsilon(e_n)] \\ &= \frac{a}{2} \sum_{i=1}^{n-1} \text{tr}[\epsilon(e_i)\iota(e_i)] - b \sum_{i=1}^{n-1} \text{tr}[\iota(e_i)\epsilon(e_i)] = 12(a-b), \end{aligned} \quad (3.19)$$

and

$$\sum_{i=1}^{n-1} \text{tr}[\epsilon(dx_n)\tilde{c}(e_i)c(e_i)c(e_n)] = 12(a+b). \quad (3.20)$$

Combining (3.17)-(3.20), we have

$$\text{tr}[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi')\epsilon(dx_n)](x_0)|_{|\xi'|=1} = 6ab^2h'(0). \quad (3.21)$$

Others are similarly. \square

Let us now turn to compute Φ (see formula (2.5) for definition of Φ). Since the sum is taken over $-r - \ell + k + j + |\alpha| = 3$, $r, \ell \leq -1$, then we have the following five cases:

Case a (I): $r = -1$, $\ell = -1$, $k = j = 0$, $|\alpha| = 1$

From (2.5) we have

$$\text{Case a (I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{\xi'}^{\alpha} \pi_{\xi_n}^{+} \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{x'}^{\alpha} \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.22)$$

Then an application of Lemma 3.2 shows that,

$$\partial_{x_i} \sigma_{-1}((\tilde{D}^*)^{-1})(x_0) = \partial_{x_i} \left(\frac{\sqrt{-1}\tilde{c}(\xi)}{ab|\xi|^2} \right) (x_0) = \frac{\sqrt{-1}\partial_{x_i}[\tilde{c}(\xi)](x_0)}{ab|\xi|^2} - \frac{\sqrt{-1}\tilde{c}(\xi)\partial_{x_i}(|\xi|^2)(x_0)}{ab|\xi|^4} = 0, \quad (3.23)$$

so Case a (I) vanishes.

Case a (II): $r = -1$, $\ell = -1$, $k = |\alpha| = 0$, $j = 1$

From (2.5) we have

$$\text{Case a (II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^{+} \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}^*)^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (3.24)$$

From Lemma 3.1 and Lemma 3.2, we have

$$\partial_{x_n} \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} = \frac{\sqrt{-1}\partial_{x_n}[\tilde{c}(\xi)](x_0)}{ab|\xi|^2} - \frac{\sqrt{-1}\tilde{c}(\xi)h'(0)}{ab|\xi|^4}. \quad (3.25)$$

By the Cauchy integral formula, we obtain

$$\pi_{\xi_n}^{+} \left[\frac{1}{(1 + \xi_n^2)^2} \right] (x_0)|_{|\xi'|=1} = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{1}{(\eta_n + i)^2(\xi_n + iu - \eta_n)} d\eta_n = -\frac{i\xi_n + 2}{4(\xi_n - i)^2}, \quad (3.26)$$

and

$$\pi_{\xi_n}^{+} \left[\frac{\sqrt{-1}\partial_{x_n}c(\xi')}{|\xi|^2} \right] (x_0)|_{|\xi'|=1} = \frac{\partial_{x_n}[c(\xi')](x_0)}{2(\xi_n - i)}. \quad (3.27)$$

Then

$$\begin{aligned}
& \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} \\
&= \frac{\partial_{x_n}[\tilde{c}(\xi')](x_0)}{2ab(\xi_n - i)} + \frac{ih'(0)}{ab} \left[\frac{i\tilde{c}(\xi')}{4(\xi_n - i)} + \frac{\tilde{c}(\xi') + i\tilde{c}(dx_n)}{4(\xi_n - i)^2} \right]. \\
&= -\frac{\partial_{x_n}[\iota(\xi')](x_0)}{2a(\xi_n - i)} + \frac{ih'(0)}{ab} \left[\frac{a(i\xi_n + 2)\epsilon(\xi')}{4(\xi_n - i)^2} - \frac{b(i\xi_n + 2)\iota(\xi')}{4(\xi_n - i)^2} + \frac{ai\epsilon(dx_n)}{4(\xi_n - i)^2} - \frac{bi\iota(dx_n)}{4(\xi_n - i)^2} \right]. \quad (3.28)
\end{aligned}$$

From Lemma 3.2, we have

$$\begin{aligned}
& \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}^*)^{-1})(x_0)|_{|\xi'|=1} \\
&= \frac{\sqrt{-1}}{ab} \left(-\frac{6\xi_n \bar{c}(dx_n) + 2\bar{c}(\xi')}{|\xi|^4} + \frac{8\xi_n^2 \bar{c}(\xi)}{|\xi|^6} \right) \\
&= \frac{\sqrt{-1}}{ab} \left[\frac{b(6\xi_n^2 - 2)\epsilon(\xi')}{(1 + \xi_n^2)^3} - \frac{a(6\xi_n^2 - 2)\iota(\xi')}{(1 + \xi_n^2)^3} + \frac{b(2\xi_n^3 - 6\xi_n)\epsilon(dx_n)}{(1 + \xi_n^2)^3} - \frac{a(2\xi_n^3 - 6\xi_n)\iota(dx_n)}{(1 + \xi_n^2)^3} \right]. \quad (3.29)
\end{aligned}$$

By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\begin{aligned}
& \text{tr}[\epsilon(\xi')\iota(\xi')] = 8; \quad \text{tr}[\epsilon(dx_n)\iota(dx_n)] = 8; \quad \text{tr}[\partial_{x_n}\iota(\xi')\epsilon(\xi')](x_0)|_{|\xi'|=1} = 8h'(0); \\
& \text{tr}[\partial_{x_n}\iota(\xi')\iota(dx_n)\epsilon(\xi')\epsilon(dx_n)](x_0)|_{|\xi'|=1} = -4h'(0); \quad \text{tr}[\partial_{x_n}\iota(\xi')\iota(\xi')\epsilon(\xi')\epsilon(dx_n)](x_0)|_{|\xi'|=1} = 0. \quad (3.30)
\end{aligned}$$

Combining (3.28), (3.29) and (3.30), we have

$$\begin{aligned}
& \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}((\tilde{D}^*)^{-1})](x_0)|_{|\xi'|=1} \\
&= \frac{h'(0)}{a^2} \frac{8(-i\xi_n - i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} + \frac{h'(0)}{b^2} \frac{8(-1 - 2i\xi_n + 3\xi_n^2 + 2i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3}. \quad (3.31)
\end{aligned}$$

Hence

$$\begin{aligned}
& \text{Case a (II)} \\
&= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left[\frac{h'(0)}{a^2} \frac{8(-i\xi_n - i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} + \frac{h'(0)}{b^2} \frac{8(-1 - 2i\xi_n + 3\xi_n^2 + 2i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} \right] d\xi_n \sigma(\xi') dx' \\
&= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \int_{\Gamma^+} \frac{8(-i\xi_n - i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n dx' - \frac{1}{2} \frac{h'(0)}{b^2} \Omega_3 \int_{\Gamma^+} \frac{8(-1 - 2i\xi_n + 3\xi_n^2 + 2i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n dx' \\
&= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \frac{2\pi i}{4!} \left[\frac{8(-i\xi_n - i\xi_n^3)}{(\xi_n + i)^3} \right]^{(4)}_{|\xi_n=i} dx' - \frac{1}{2} \frac{h'(0)}{b^2} \Omega_3 \frac{2\pi i}{4!} \left[\frac{8(-1 - 2i\xi_n + 3\xi_n^2 + 2i\xi_n^3)}{(\xi_n + i)^3} \right]^{(4)}_{|\xi_n=i} dx' \\
&= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \frac{2\pi i}{4!} (-12i) dx' - \frac{1}{2} \frac{h'(0)}{b^2} \Omega_3 \frac{2\pi i}{4!} (-24i) dx' \\
&= \left(-\frac{1}{2a^2} - \frac{1}{b^2} \right) \pi h'(0) \Omega_3 dx', \quad (3.32)
\end{aligned}$$

where Ω_3 is the canonical volume of S^3 .

Case a (III): $r = -1$, $\ell = -1$, $j = |\alpha| = 0$, $k = 1$

From (2.5) we have

$$\text{Case a (III)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\tilde{D}^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.33)$$

From Lemma 3.1 and Lemma 3.2, we get

$$\begin{aligned}
\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} &= -\frac{\tilde{c}(\xi') + i\tilde{c}(dx_n)}{2ab(\xi_n - i)^2} \\
&= -\frac{a\epsilon(\xi') - b\iota(\xi') + i(\epsilon(dx_n) + \iota(dx_n))}{2ab(\xi_n - i)^2}, \quad (3.34)
\end{aligned}$$

and

$$\begin{aligned}
& \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\tilde{D}^*)^{-1}(x_0))|_{|\xi'|=1} \\
&= \frac{-\sqrt{-1}h'(0)}{ab} \left[\frac{\bar{c}(dx_n)}{|\xi|^4} - 4\xi_n \frac{\bar{c}(\xi') + \xi_n \bar{c}(dx_n)}{|\xi|^6} \right] - \frac{2\xi_n \sqrt{-1} \partial_{x_n} \bar{c}(\xi')(x_0)}{ab|\xi|^4} \\
&= \frac{2i\xi_n \partial_{x_n} [\iota(\xi')](x_0)}{b|\xi|^4} + \frac{ih'(0)}{ab} \left[\frac{4b\xi_n \epsilon(\xi')}{|\xi|^6} - \frac{4a\xi_n \iota(\xi')}{|\xi|^6} - \frac{b(|\xi|^2 - 4\xi_n^2) \epsilon(dx_n)}{|\xi|^6} + \frac{a(|\xi|^2 - 4\xi_n^2) \iota(dx_n)}{|\xi|^6} \right].
\end{aligned} \tag{3.35}$$

Combining (3.34) and (3.35), we obtain

$$\begin{aligned}
& \text{trace} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}((\tilde{D}^*)^{-1}) \right] (x_0)|_{|\xi'|=1} \\
&= \frac{h'(0)}{a^2} \frac{4(1 + 4i\xi_n - 3\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} - \frac{h'(0)}{b^2} \frac{4(-1 - 2i\xi_n + 3\xi_n^2 + 2i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3}.
\end{aligned} \tag{3.36}$$

Then

$$\begin{aligned}
& \text{Case a (III)} \\
&= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \left[\frac{h'(0)}{a^2} \frac{4(1 + 4i\xi_n - 3\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} + \frac{h'(0)}{b^2} \frac{4(-1 - 2i\xi_n + 3\xi_n^2 + 2i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} \right] d\xi_n \sigma(\xi') dx' \\
&= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \int_{\Gamma^+} \frac{4(1 + 4i\xi_n - 3\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n dx' + \frac{1}{2} \frac{h'(0)}{b^2} \Omega_3 \int_{\Gamma^+} \frac{4(-1 - 2i\xi_n + 3\xi_n^2 + 2i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n dx' \\
&= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \frac{2\pi i}{4!} (-12i) dx' + \frac{1}{2} \frac{h'(0)}{b^2} \Omega_3 \frac{2\pi i}{4!} (24i) dx' \\
&= \left(\frac{1}{a^2} + \frac{1}{2b^2} \right) \pi h'(0) \Omega_3 dx'.
\end{aligned} \tag{3.37}$$

Case b: $r = -2$, $\ell = -1$, $k = j = |\alpha| = 0$

From (2.5) we have

$$\text{Case b} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \tag{3.38}$$

By Lemma 3.1 and Lemma 3.2, we obtain

$$\begin{aligned}
& \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1})(x_0) \\
&= \frac{\sqrt{-1}}{ab} \left(-\frac{2\xi_n^2 \bar{c}(dx_n) + 2\xi_n \bar{c}(\xi')}{|\xi|^4} + \frac{\bar{c}(dx_n)}{|\xi|^2} \right) \\
&= \frac{\sqrt{-1}}{ab} \left[\frac{-2b\xi_n \epsilon(\xi')}{(1 + \xi_n^2)^2} + \frac{2a\xi_n \iota(\xi')}{(1 + \xi_n^2)^2} + \frac{b(1 - \xi_n^2) \epsilon(dx_n)}{(1 + \xi_n^2)^2} + \frac{a(\xi_n^2 - 1) \iota(dx_n)}{(1 + \xi_n^2)^2} \right]
\end{aligned} \tag{3.39}$$

and

$$\sigma_{-2}(\tilde{D}^{-1})(x_0) = \frac{\tilde{c}(\xi) \tilde{p}_0(x_0) \tilde{c}(\xi)}{a^2 b^2 |\xi|^4} + \frac{\tilde{c}(\xi)}{a^2 b^2 |\xi|^6} \tilde{c}(dx_n) \left[\partial_{x_n} [\tilde{c}(\xi')](x_0) |\xi|^2 - \tilde{c}(\xi) h'(0) |\xi|_{\partial M}^2 \right]. \tag{3.40}$$

Then

$$\begin{aligned}
& \pi_{\xi_n}^+ \sigma_{-2}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} \\
&= \pi_{\xi_n}^+ \left[\frac{\tilde{c}(\xi) \tilde{p}_0(x_0) \tilde{c}(\xi) + \tilde{c}(\xi) \tilde{c}(dx_n) \partial_{x_n} [\tilde{c}(\xi')](x_0)}{a^2 b^2 (1 + \xi_n^2)^2} \right] - h'(0) \pi_{\xi_n}^+ \left[\frac{\tilde{c}(\xi) \tilde{c}(dx_n) \tilde{c}(\xi)}{a^2 b^2 (1 + \xi_n^2)^3} \right] \\
&:= B_1 - B_2,
\end{aligned} \tag{3.41}$$

where

$$\begin{aligned}
B_1 &= \frac{-1}{4a^2b^2(\xi_n - i)^2} \left[(2 + i\xi_n)\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi') + i\xi_n\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n) \right. \\
&\quad \left. + (2 + i\xi_n)\tilde{c}(\xi')\tilde{c}(dx_n)\partial_{x_n}\tilde{c}(\xi') + i\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi') + i\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n) - i\partial_{x_n}\tilde{c}(\xi') \right] \\
&= \frac{-1}{4a^2b^2(\xi_n - i)^2} \left[(2 + i\xi_n)\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi') + i\xi_n\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n) + i\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi') + i\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n) \right] \\
&\quad + \frac{(2 + i\xi_n)\epsilon(\xi')\epsilon(dx_n)\partial_{x_n}[\iota(\xi')](x_0)}{4b(\xi_n - i)^2} + \frac{b(2 + i\xi_n)\iota(\xi')\iota(dx_n)\partial_{x_n}[\iota(\xi')](x_0)}{4a^2(\xi_n - i)^2} - \frac{i\partial_{x_n}[\iota(\xi')](x_0)}{4a(\xi_n - i)^2} \quad (3.42)
\end{aligned}$$

From (3.39) and (3.42), we have

$$\begin{aligned}
&\text{trace} \left[B_1 \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1}) \right] (x_0) |_{|\xi'|=1} \\
&= \frac{h'(0)}{a^2} \times \frac{-i(3 + 5i\xi_n - 3\xi_n^2)}{(\xi_n - i)^2(1 + \xi_n^2)^2} + \frac{h'(0)}{b^2} \times \frac{i(-1 - 23i\xi_n + \xi_n^2 + 2i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^2}. \quad (3.43)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
B_2 &= h'(0)\pi_{\xi_n}^+ \left[\frac{-\xi_n^2 c(dx_n)^2 - 2\xi_n c(\xi') + c(dx_n)}{a^2b^2(1 + \xi_n^2)^3} \right] \\
&= \frac{h'(0)}{2ab} \left[\frac{\tilde{c}(dx_n)}{4i(\xi_n - i)} + \frac{\tilde{c}(dx_n) - i\tilde{c}(\xi')}{8(\xi_n - i)^2} + \frac{3\xi_n - 7i}{8(\xi_n - i)^3} [i\tilde{c}(\xi') - \tilde{c}(dx_n)] \right] \\
&= \frac{(3 + i\xi_n)h'(0)\epsilon(\xi')}{8b(\xi_n - i)^3} - \frac{(3 + i\xi_n)h'(0)\iota(\xi')}{8a(\xi_n - i)^3} + \frac{(-i\xi_n^2 - 3\xi_n + 4i)h'(0)\epsilon(dx_n)}{8b(\xi_n - i)^3} \\
&\quad - \frac{(-i\xi_n^2 - 3\xi_n + 4i)h'(0)\iota(dx_n)}{8a(\xi_n - i)^3}. \quad (3.44)
\end{aligned}$$

From (3.39) and (3.44), we obtain

$$\begin{aligned}
&\text{trace} \left[B_2 \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1}) \right] (x_0) |_{|\xi'|=1} \\
&= \frac{h'(0)}{a^2} \frac{-i(4i - 9\xi_n - 7i\xi_n^2 + 3\xi_n^2 + i\xi_n^4)}{(\xi_n - i)^3(1 + \xi_n^2)^2} + \frac{h'(0)}{b^2} \frac{-i(4i - 9\xi_n - 7i\xi_n^2 + 3\xi_n^2 + i\xi_n^4)}{(\xi_n - i)^3(1 + \xi_n^2)^2}. \quad (3.45)
\end{aligned}$$

Combining (3.43) and (3.45), we have

$$\begin{aligned}
\text{case b)} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[(B_1 - B_2) \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx' \\
&= \left(-\frac{1}{8a^2} + \frac{11}{8b^2} \right) \pi h'(0) \Omega_3 dx'. \quad (3.46)
\end{aligned}$$

Case c: $r = -1$, $\ell = -2$, $k = j = |\alpha| = 0$

From(2.5) we have

$$\text{Case c} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_{-2}((\tilde{D}^*)^{-1})] (x_0) d\xi_n \sigma(\xi') dx'. \quad (3.47)$$

By Lemma 3.1, Lemma 3.2 and Lemma 3.4, we obtain

$$\begin{aligned}
\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1})(x_0) |_{|\xi'|=1} &= \frac{\tilde{c}(\xi') + i\tilde{c}(dx_n)}{2ab(\xi_n - i)} \\
&= \frac{\epsilon(\xi')}{2b(\xi_n - i)} - \frac{\iota(\xi')}{2a(\xi_n - i)} + \frac{i\epsilon(dx_n)}{2b(\xi_n - i)} - \frac{i\iota(dx_n)}{2a(\xi_n - i)} \quad (3.48)
\end{aligned}$$

and

$$\begin{aligned}
& \partial_{\xi_n} \sigma_{-2}((\tilde{D}^*)^{-1})(x_0) \\
= & \frac{1}{a^2 b^2 (1 + \xi_n^2)^3} \left[(2\xi_n - 2\xi_n^3) \bar{c}(dx_n) \bar{p}_0 \bar{c}(dx_n) + (1 - 3\xi_n^2) \bar{c}(dx_n) \bar{p}_0 \bar{c}(\xi') \right. \\
& + (1 - 3\xi_n^2) \bar{c}(\xi') \bar{p}_0 \bar{c}(dx_n) - 4\xi_n \bar{c}(\xi') \bar{p}_0 \bar{c}(\xi') + (3\xi_n^2 - 1) ab \partial_{x_n} \bar{c}(\xi') - 4\xi_n \bar{c}(\xi') \bar{c}(dx_n) \partial_{x_n} \bar{c}(\xi') \\
& \left. + 2ab h'(0) \bar{c}(\xi') + 2abh'(0) \xi_n \bar{c}(dx_n) \right] + 6\xi_n h'(0) \frac{\bar{c}(\xi) \bar{c}(dx_n) \bar{c}(\xi)}{a^2 b^2 (1 + \xi_n^2)^4} \\
= & \frac{1}{a^2 b^2 (1 + \xi_n^2)^3} \left[(2\xi_n - 2\xi_n^3) \bar{c}(dx_n) \bar{p}_0 \bar{c}(dx_n) + (1 - 3\xi_n^2) \bar{c}(dx_n) \bar{p}_0 \bar{c}(\xi') \right. \\
& \left. + (1 - 3\xi_n^2) \bar{c}(\xi') \bar{p}_0 \bar{c}(dx_n) - 4\xi_n \bar{c}(\xi') \bar{p}_0 \bar{c}(\xi') \right] \\
& + \frac{(2 - 10\xi_n^2) h'(0) \epsilon(\xi')}{a(1 + \xi_n^2)^4} - \frac{(2 - 10\xi_n^2) h'(0) \iota(\xi')}{b(1 + \xi_n^2)^4} + \frac{(8\xi_n - 4\xi_n^3) h'(0) \epsilon(dx_n)}{2a(1 + \xi_n^2)^4} \\
& - \frac{(8\xi_n - 4\xi_n^3) h'(0) \iota(dx_n)}{b(1 + \xi_n^2)^4} - \frac{3(\xi_n^2 - 1) \partial_{x_n} [\iota(\xi')](x_0)}{b(1 + \xi_n^2)^3} \\
& + \frac{4\xi_n \epsilon(\xi') \epsilon(dx_n) \partial_{x_n} [\iota(\xi')](x_0)}{a(1 + \xi_n^2)^3} + \frac{4a\xi_n \iota(\xi') \iota(dx_n) \partial_{x_n} [\iota(\xi')](x_0)}{b^2(1 + \xi_n^2)^3}. \tag{3.49}
\end{aligned}$$

Combining (3.48) and (3.49), we have

$$\begin{aligned}
& \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_{-2}((\tilde{D}^*)^{-1})(x_0)]|_{|\xi'|=1} \\
= & \frac{h'(0)}{a^2} \frac{-2(-7 - 21i\xi_n + 26\xi_n^2 + 6i\xi_n^3 + 9\xi_n^4 + 3i\xi_n^5)}{(\xi_n - i)(1 + \xi_n^2)^4} + \frac{h'(0)}{b^2} \frac{-2(-1 - 25i\xi_n + 26\xi_n^2 + 2i\xi_n^3 + 3\xi_n^4 + 3i\xi_n^5)}{(\xi_n - i)(1 + \xi_n^2)^4}. \tag{3.50}
\end{aligned}$$

Then similarly to computations of the case b), we have

$$\text{case c)} = \frac{5}{2a^2} \pi h'(0) \Omega_3 dx'. \tag{3.51}$$

Since Φ is the sum of the **case a**, **b** and **c**,

$$\Phi = \frac{23}{8} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \pi h'(0) \Omega_3 dx'. \tag{3.52}$$

Then we have

Theorem 3.5. *Let M be a four dimensional compact connected manifold with the boundary ∂M and the metric g^M as above, and \tilde{D}, \tilde{D}^* are the nonminimal de Rham-Hodge operators on $C^\infty(\Lambda^*(T^*M))$, then*

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = 4\pi \int_M \sum_{k=0}^4 c_1(4, k, a, b) R d\text{vol}(M) - \frac{23}{12} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) \pi \int_{\partial M} K \Omega_3 dx', \tag{3.53}$$

where R is the scalar curvature and $c_1(4, k, a, b) = b^{-2} \{ \frac{1}{6} \binom{4}{k} - \binom{2}{k-1} \} + (b^{-2} - a^{-2}) \sum_{j < k} (-1)^{j-k} \{ \frac{1}{6} \binom{4}{j} - \binom{2}{j-1} \}$.

Let us now consider the Einstein-Hilbert action for four dimensional manifolds with boundary. Recall the Einstein-Hilbert action for manifolds with boundary[12],

$$I_{\text{Gr}} = \frac{1}{16\pi} \int_M R d\text{vol}_M + 2 \int_{\partial M} K d\text{vol}_{\partial M} := I_{\text{Gr},i} + I_{\text{Gr},b}, \tag{3.54}$$

where

$$K = \sum_{1 \leq i, j \leq n-1} K_{i,j} g_{\partial M}^{i,j}; \quad K_{i,j} = -\Gamma_{i,j}^n, \quad (3.55)$$

and $K_{i,j}$ is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, $K_{i,j}(x_0) = -\Gamma_{i,j}^n(x_0) = -\frac{1}{2}h'(0)$, when $i = j < n$, otherwise is zero.

Let

$$\widetilde{\text{Wres}}[\pi^+(\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = \widetilde{\text{Wres}}_i[\pi^+\tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] + \widetilde{\text{Wres}}_b[\pi^+\tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}], \quad (3.56)$$

where

$$\widetilde{\text{Wres}}_i[\pi^+(\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = \int_M \int_{|\xi|=1} \text{trace}_{\Lambda^*(T^*M)}[\sigma_{-4}((\tilde{D}\tilde{D}^*)^{-1})]\sigma(\xi)dx \quad (3.57)$$

and

$$\begin{aligned} & \widetilde{\text{Wres}}_b[\pi^+(\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] \\ &= \int_{\partial M} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{j,k=0}^{\infty} \sum \frac{(-i)^{|\alpha|+j+k+1}}{\alpha!(j+k+1)!} \times \text{trace}_{\Lambda^*(T^*M)}[\partial_{x_n}^j \partial_{\xi'}^\alpha \partial_{\xi_n}^k \sigma_r^+((\tilde{D}^{-1})(x', 0, \xi', \xi_n) \\ & \quad \times \partial_{x'}^\alpha \partial_{\xi_n}^{j+1} \partial_{x_n}^k \sigma_l((\tilde{D}^*)^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx' \end{aligned} \quad (3.58)$$

denote the interior term and boundary term of $\widetilde{\text{Wres}}[\pi^+(\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}]$. Combining (3.42), (3.43) and (3.45), we obtain

Theorem 3.6. *Let M be a four dimensional compact manifold with the boundary ∂M associated to non-minimal de Rham-Hodge operators \tilde{D} and \tilde{D}^* . Assume ∂M is flat, then*

$$\begin{aligned} I_{\text{Gr},i} &= \frac{1}{64\pi c_1(4, k, a, b)} \widetilde{\text{Wres}}_i[\pi^+(\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}]; \\ I_{\text{Gr},b} &= \frac{-24}{23(\frac{1}{a^2} + \frac{1}{b^2})\pi\Omega_3} \widetilde{\text{Wres}}_b[\pi^+(\tilde{D})^{-1} \circ \pi^+(\tilde{D}^*)^{-1}]. \end{aligned} \quad (3.59)$$

4. The Kastler-Kalau-Walze type theorem of the nonminimal de Rham-Hodge operators \tilde{D}

In this section, we compute the lower dimension volume for four dimension compact connected manifolds with boundary associated to nonminimal de Rham-Hodge operators \tilde{D} and get a Kastler-Kalau-Walze type theorem in this case. Let M be an four dimensional compact oriented connected manifold with boundary ∂M , and the metric g^M on M as above. Note that $[\sigma_{-4}(\tilde{D}^{-2})]|_M$ has the same expression with the case of without boundary in [12], so locally we can use Theorem 3.1 in [12] to compute the first term. Therefore

$$\int_M \int_{|\xi|=1} \text{trace}_{\Lambda^*(T^*M)}[\sigma_{-4}(\tilde{D}^{-2})]\sigma(\xi)dx = \frac{8\Omega_4}{3ab} \int_M R \text{dvol}_M, \quad (4.1)$$

where R is the scalar curvature.

Let us now turn to compute Φ (see formula (2.5) for definition of Φ). Since the sum is taken over $-r - \ell + k + j + |\alpha| = 3$, $r, \ell \leq -1$, then we have the following five cases:

Case a (I): $r = -1$, $\ell = -1$, $k = j = 0$, $|\alpha| = 1$

From (2.5) we have

$$\text{Case a (I)} = - \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \sum_{|\alpha|=1} \text{trace} \left[\partial_{\xi'}^\alpha \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{x'}^\alpha \partial_{\xi_n} \sigma_{-1}(\tilde{D}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (4.2)$$

Then an application of Lemma 3.2 shows that,

$$\partial_{x_i} \sigma_{-1}(\tilde{D}^{-1})(x_0) = \partial_{x_i} \left(\frac{\sqrt{-1} \tilde{c}(\xi)}{ab|\xi|^2} \right) (x_0) = \frac{\sqrt{-1} \partial_{x_i} [\tilde{c}(\xi)](x_0)}{ab|\xi|^2} - \frac{\sqrt{-1} \tilde{c}(\xi) \partial_{x_i} (|\xi|^2)(x_0)}{ab|\xi|^4} = 0, \quad (4.3)$$

so Case a (I) vanishes.

Case a (II): $r = -1$, $\ell = -1$, $k = |\alpha| = 0$, $j = 1$

From (2.5) we have

$$\text{Case a (II)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}^{-1})](x_0) d\xi_n \sigma(\xi') dx'. \quad (4.4)$$

By Lemma 3.1, Lemma 3.2, we have

$$\partial_{x_n} \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} = \frac{\sqrt{-1} \partial_{x_n} [\tilde{c}(\xi)](x_0)}{ab|\xi|^2} - \frac{\sqrt{-1} \tilde{c}(\xi) h'(0)}{ab|\xi|^4}. \quad (4.5)$$

By the Cauchy integral formula we obtain

$$\pi_{\xi_n}^+ \left[\frac{1}{(1 + \xi_n^2)^2} \right] (x_0)|_{|\xi'|=1} = \frac{1}{2\pi i} \lim_{u \rightarrow 0^-} \int_{\Gamma^+} \frac{\frac{1}{(\eta_n + i)^2 (\xi_n + iu - \eta_n)}}{(\eta_n - i)^2} d\eta_n = -\frac{i\xi_n + 2}{4(\xi_n - i)^2}, \quad (4.6)$$

and

$$\pi_{\xi_n}^+ \left[\frac{\sqrt{-1} \partial_{x_n} c(\xi')}{|\xi|^2} \right] (x_0)|_{|\xi'|=1} = \frac{\partial_{x_n} [c(\xi')](x_0)}{2(\xi_n - i)}. \quad (4.7)$$

Then

$$\begin{aligned} & \partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} \\ &= \frac{\partial_{x_n} [\tilde{c}(\xi')](x_0)}{2ab(\xi_n - i)} + \frac{ih'(0)}{ab} \left[\frac{i\tilde{c}(\xi')}{4(\xi_n - i)} + \frac{\tilde{c}(\xi') + i\tilde{c}(dx_n)}{4(\xi_n - i)^2} \right]. \\ &= -\frac{\partial_{x_n} [\iota(\xi')](x_0)}{2a(\xi_n - i)} + \frac{ih'(0)}{ab} \left[\frac{a(i\xi_n + 2)\epsilon(\xi')}{4(\xi_n - i)^2} - \frac{b(i\xi_n + 2)\iota(\xi')}{4(\xi_n - i)^2} + \frac{ai\epsilon(dx_n)}{4(\xi_n - i)^2} - \frac{bi\iota(dx_n)}{4(\xi_n - i)^2} \right]. \end{aligned} \quad (4.8)$$

By Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned} & \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} \\ &= \frac{\sqrt{-1}}{ab} \left(-\frac{6\xi_n \tilde{c}(dx_n) + 2\tilde{c}(\xi')}{|\xi|^4} + \frac{8\xi_n^2 \tilde{c}(\xi)}{|\xi|^6} \right) \\ &= \frac{i(6\xi_n^2 - 2)\epsilon(\xi')}{b(1 + \xi_n^2)^3} - \frac{i(6\xi_n^2 - 2)\iota(\xi')}{a(1 + \xi_n^2)^3} + \frac{i(2\xi_n^3 - 6\xi_n)\epsilon(dx_n)}{b(1 + \xi_n^2)^3} - \frac{i(2\xi_n^3 - 6\xi_n)\iota(dx_n)}{a(1 + \xi_n^2)^3}. \end{aligned} \quad (4.9)$$

Combining (4.5) and (4.9), we have

$$\text{trace}[\partial_{x_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(\tilde{D}^{-1})](x_0)|_{|\xi'|=1} = \frac{h'(0)}{ab} \frac{8(-1 - 3i\xi_n + 3\xi_n^2 + i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3}. \quad (4.10)$$

Hence

$$\begin{aligned} \text{Case a (II)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{h'(0)}{ab} \frac{8(-1 - 3i\xi_n + 3\xi_n^2 + i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \frac{h'(0)}{ab} \Omega_3 \int_{\Gamma^+} \frac{8(-1 - 3i\xi_n + 3\xi_n^2 + i\xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n dx' \\ &= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \frac{2\pi i}{4!} \left[\frac{8(-1 - 3i\xi_n + 3\xi_n^2 + i\xi_n^3)}{(\xi_n + i)^3} \right]^{(4)} \Big|_{\xi_n=i} dx' \\ &= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \frac{2\pi i}{4!} (-36i) dx' \\ &= -\frac{3}{2ab} \pi h'(0) \Omega_3 dx'. \end{aligned} \quad (4.11)$$

Case a (III): $r = -1$, $\ell = -1$, $j = |\alpha| = 0$, $k = 1$

From (2.5) we have

$$\text{Case a (III)} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}^{-1}) \right] (x_0) d\xi_n \sigma(\xi') dx'. \quad (4.12)$$

From Lemma 3.1 and Lemma 3.2 we obtain

$$\begin{aligned} \partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} &= -\frac{\tilde{c}(\xi') + i\tilde{c}(dx_n)}{2ab(\xi_n - i)^2} \\ &= -\frac{a\epsilon(\xi') - b\iota(\xi') + i(\epsilon(dx_n) + \iota(dx_n))}{2ab(\xi_n - i)^2}, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} &\partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}^{-1})(x_0)|_{|\xi'|=1} \\ &= \frac{-\sqrt{-1}h'(0)}{ab} \left[\frac{\tilde{c}(dx_n)}{|\xi|^4} - 4\xi_n \frac{\tilde{c}(\xi') + \xi_n \tilde{c}(dx_n)}{|\xi|^6} \right] - \frac{2\xi_n \sqrt{-1} \partial_{x_n} \tilde{c}(\xi')(x_0)}{ab|\xi|^4} \\ &= \frac{2i\xi_n \partial_{x_n} [\iota(\xi')](x_0)}{a(1 + \xi_n^2)^2} + \frac{4i\xi_n h'(0)\epsilon(\xi')}{b(1 + \xi_n^2)^3} - \frac{4i\xi_n h'(0)\iota(\xi')}{a(1 + \xi_n^2)^3} + \frac{(3\xi_n^2 - 1)ih'(0)\epsilon(dx_n)}{b(1 + \xi_n^2)^3} - \frac{(3\xi_n^2 - 1)ih'(0)\iota(dx_n)}{a(1 + \xi_n^2)^3}. \end{aligned} \quad (4.14)$$

From (4.13) and (4.14), we have

$$\text{trace} \left[\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma_{-1}(\tilde{D}^{-1}) \right] (x_0)|_{|\xi'|=1} = \frac{h'(0)}{ab} \frac{8i(-i + 3\xi_n + 3i\xi_n^2 - \xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3}. \quad (4.15)$$

Then

$$\begin{aligned} \text{Case a (III)} &= -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{h'(0)}{ab} \frac{8i(-i + 3\xi_n + 3i\xi_n^2 - \xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n \sigma(\xi') dx' \\ &= -\frac{1}{2} \frac{h'(0)}{ab} \Omega_3 \int_{\Gamma^+} \frac{8i(-i + 3\xi_n + 3i\xi_n^2 - \xi_n^3)}{(\xi_n - i)^2(1 + \xi_n^2)^3} d\xi_n dx' \\ &= -\frac{1}{2} \frac{h'(0)}{a^2} \Omega_3 \frac{2\pi i}{4!} (36i) dx' \\ &= \frac{3}{2ab} \pi h'(0) \Omega_3 dx'. \end{aligned} \quad (4.16)$$

Case b: $r = -1$, $\ell = -2$, $k = j = |\alpha| = 0$

From (2.5) and the Leibniz rule, we obtain

$$\begin{aligned} \text{Case b} &= -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_{-2}(\tilde{D}^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \\ &= i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace} [\partial_{\xi_n} \pi_{\xi_n}^+ \sigma_{-1}(\tilde{D}^{-1}) \times \sigma_{-2}(\tilde{D}^{-1})] (x_0) d\xi_n \sigma(\xi') dx' \end{aligned} \quad (4.17)$$

From Lemma 3.1, Lemma 3.2 and Lemma 3.4 we obtain

$$\begin{aligned}
& \sigma_{-2}(\tilde{D}^{-1})(x_0) \\
&= \frac{\tilde{c}(\xi)\tilde{p}_0(x_0)\tilde{c}(\xi)}{a^2b^2|\xi|^4} + \frac{\tilde{c}(\xi)}{a^2b^2|\xi|^6}\tilde{c}(dx_n)\left[\partial_{x_n}[\tilde{c}(\xi')](x_0)|\xi|^2 - \tilde{c}(\xi)h'(0)|\xi|_{\partial M}^2\right] \\
&= \frac{1}{a^2b^2(1+\xi_n^2)^2}\left[\tilde{c}(\xi')\tilde{p}_0\tilde{c}(\xi') + \xi_n\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(\xi') + \xi_n\tilde{c}(\xi')\tilde{p}_0\tilde{c}(dx_n) + \xi_n^2\tilde{c}(dx_n)\tilde{p}_0\tilde{c}(dx_n)\right] \\
&\quad + \frac{2\xi_nh'(0)\epsilon(\xi')}{b(1+\xi_n^2)^3} - \frac{2\xi_nh'(0)\iota(\xi')}{a(1+\xi_n^2)^3} + \frac{(\xi_n^2-1)h'(0)\epsilon(dx_n)}{b(1+\xi_n^2)^3} \\
&\quad + \frac{(1-\xi_n^2)h'(0)\iota(dx_n)}{a(1+\xi_n^2)^3} + \frac{\xi_n\partial_{x_n}[\iota(\xi')](x_0)}{a(1+\xi_n^2)^2} \\
&\quad + \frac{\epsilon(\xi')\epsilon(dx_n)\partial_{x_n}[\iota(\xi')](x_0)}{b(1+\xi_n^2)^2} - \frac{b\xi_n\iota(\xi')\iota(dx_n)\partial_{x_n}[\iota(\xi')](x_0)}{a^2(1+\xi_n^2)^2}.
\end{aligned} \tag{4.18}$$

From (4.13) and (4.18), we obtain

$$\text{trace}[\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}^{-1})\times\sigma_{-2}(\tilde{D}^{-1})](x_0)|_{|\xi'|=1} = \frac{h'(0)}{ab}\frac{2(-8i+12\xi_n+3i\xi_n^2+4\xi_n^3+3i\xi_n^4)}{(\xi_n-i)^2(1+\xi_n^2)^3}.$$

Then similarly to computations of the case a), we have

$$\text{case b)} = -\frac{6}{ab}\pi h'(0)\Omega_3 dx'.$$

Case c: $r = -2$, $\ell = -1$, $k = j = |\alpha| = 0$

From (2.5) we have

$$\text{Case c} = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+\sigma_{-2}(\tilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-1})](x_0)d\xi_n\sigma(\xi')dx'. \tag{4.20}$$

By the Leibniz rule, trace property and "+" and "-" vanishing after the integration over ξ_n [8][18], then

$$\begin{aligned}
& \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+\sigma_{-2}(\tilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-2}(\tilde{D}^{-2})]d\xi_n \\
&= \int_{-\infty}^{+\infty} \text{tr}[\sigma_{-2}(\tilde{D}^{-2})\times\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-1})]d\xi_n - \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^-\sigma_{-2}(\tilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-1})]d\xi_n \\
&= \int_{-\infty}^{+\infty} \text{tr}[\sigma_{-2}(\tilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-2})]d\xi_n - \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^-\sigma_{-2}(\tilde{D}^{-2})\times\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}^{-1})]d\xi_n \\
&= \int_{-\infty}^{+\infty} \text{tr}[\sigma_{-2}(\tilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-1})]d\xi_n - \int_{-\infty}^{+\infty} \text{tr}[\sigma_{-2}(\tilde{D}^{-1})\times\partial_{\xi_n}\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}^{-1})]d\xi_n \\
&= \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-1})\times\sigma_{-2}(\tilde{D}^{-1})]d\xi_n + \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}\sigma_{-2}(\tilde{D}^{-2})\times\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}^{-1})]d\xi_n \\
&= \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-1})\times\sigma_{-2}(\tilde{D}^{-2})]d\xi_n + \int_{-\infty}^{+\infty} \text{tr}[\pi_{\xi_n}^+\sigma_{-1}(\tilde{D}^{-1})\times\partial_{\xi_n}\sigma_{-2}(\tilde{D}^{-1})]d\xi_n.
\end{aligned}$$

Then we have

$$\text{case c)} = \text{case b)} - i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n}\sigma_{-1}(\tilde{D}^{-1})\times\sigma_{-2}(\tilde{D}^{-1})]d\xi_n\sigma(\xi')dx'. \tag{4.21}$$

By Lemma 3.1, Lemma 3.2, we obtain

$$\begin{aligned}
& \partial_{\xi_n} \sigma_{-1}(\tilde{D}^{-1})(x_0) \\
&= \frac{\sqrt{-1}}{ab} \left(-\frac{2\xi_n^2 \tilde{c}(dx_n) + 2\xi_n \tilde{c}(\xi')}{|\xi|^4} + \frac{\tilde{c}(dx_n)}{|\xi|^2} \right) \\
&= \frac{-2i\xi_n \epsilon(\xi')}{b(1 + \xi_n^2)^2} + \frac{2i\xi_n \iota(\xi')}{2(1 + \xi_n^2)^2} + \frac{i(1 - \xi_n^2) \epsilon(dx_n)}{b(1 + \xi_n^2)^2} + \frac{i(\xi_n^2 - 1) \iota(dx_n)}{a(1 + \xi_n^2)^2}.
\end{aligned} \tag{4.22}$$

From (4.18) and (4.22), we have

$$-i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_n} \sigma_{-1}(\tilde{D}^{-1}) \times \sigma_{-2}(\tilde{D}^{-1})] d\xi_n \sigma(\xi') dx' = \frac{12}{ab} \pi h'(0) \Omega_3 dx'. \tag{4.23}$$

By (4.19), (4.21) and (4.23), we have

$$\text{case c) } = \frac{6}{ab} \pi h'(0) \Omega_3 dx'. \tag{4.24}$$

Since Φ is the sum of the **case a**, **b** and **c**, so is zero. Then we have

Theorem 4.1. *Let M be a four dimensional compact connected manifold with the boundary ∂M and the metric g^M as above, and \tilde{D} are the nonminimal de Rham-Hodge operators on $C^\infty(\Lambda^*(T^*M))$, then*

$$\widetilde{\text{Wres}}[(\pi^+ \tilde{D}^{-1})^2] = \frac{8\Omega_4}{3ab} \int_M R d\text{vol}_M, \tag{4.25}$$

where R is the scalar curvature and Ω_4 is the canonical volume of S^3 .

5. Lower dimensional volumes for three dimensional spin manifolds with boundary

For an odd dimensional manifolds with boundary, as in Section 5-7 in [10], we have the formula

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}^{-1} \circ \pi^+ (\tilde{D}^*)^{-1}] = \int_{\partial M} \Phi. \tag{5.1}$$

When $n = 3$, then in (2.5), $r - k - |\alpha| + l - j - 1 = -3$, $r, l \leq -1$, so we get $r = l = -1$, $k = |\alpha| = j = 0$,

$$\Phi = \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}_{S(TM)} [\sigma_{-1}^+(\tilde{D}^{-1})(x', 0, \xi', \xi_n) \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1})(x', 0, \xi', \xi_n)] d\xi_3 \sigma(\xi') dx'. \tag{5.2}$$

By Lemma 3.1 and Lemma 3.2, we have

$$\begin{aligned}
\sigma_{-1}^+(\tilde{D}^{-1})|_{|\xi'|=1}(x_0)|_{|\xi'|=1} &= \frac{\tilde{c}(\xi') + i\tilde{c}(dx_n)}{2ab(\xi_n - i)} \\
&= \frac{a\epsilon(\xi') - b\iota(\xi') + i(a\epsilon(dx_n) - b\iota(dx_n))}{2ab(\xi_n - i)},
\end{aligned} \tag{5.3}$$

and

$$\begin{aligned}
\partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1})(x_0)|_{|\xi'|=1} &= \frac{\sqrt{-1}\tilde{c}(dx_n)}{ab(1 + \xi_n^2)} - \frac{2\sqrt{-1}\xi_n \tilde{c}(\xi)}{ab(1 + \xi_n^2)^2} \\
&= \frac{\sqrt{-1}(b\epsilon(dx_n) - a\iota(dx_n))}{ab(1 + \xi_n^2)} - \frac{2\sqrt{-1}\xi_n(b\epsilon(\xi) - a\iota(\xi))}{ab(1 + \xi_n^2)^2}.
\end{aligned} \tag{5.4}$$

For $n = 3$, we take the coordinates as in Section 2. Locally $S(TM)|_{\tilde{U}} \cong \tilde{U} \times \wedge_{\mathbb{C}}^{\text{even}}(2)$. Let $\{\tilde{f}_1, \tilde{f}_2\}$ be an orthonormal basis of $\wedge_{\mathbb{C}}^{\text{even}}(2)$ and we will compute the trace under this basis. By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, then we have the equalities:

$$\text{tr}[\epsilon(\xi')\iota(\xi')] = 4; \quad \text{tr}[\epsilon(dx_n)\iota(dx_n)] = 4. \quad (5.5)$$

Form (5.3) (5.4) and (5.5), we get

$$\text{trace}[\sigma_{-1}^+(\tilde{D}^{-1}) \times \partial_{\xi_n} \sigma_{-1}((\tilde{D}^*)^{-1})](x_0)|_{|\xi'|=1} = \frac{1}{a^2} \frac{-2\xi_n^3 + 2\xi_n + 4i\xi_n}{(\xi_n - i)(1 + \xi_n^2)^2} + \frac{1}{b^2} \frac{-2\xi_n^3 + 2\xi_n + 4i\xi_n}{(\xi_n - i)(1 + \xi_n^2)^2}. \quad (5.6)$$

By (5.2) and (5.6) and the Cauchy integral formula, we get

$$\Phi = (-\frac{1}{2} + \frac{i}{2})(\frac{1}{a^2} + \frac{1}{b^2})\pi\Omega_2\text{vol}_{\partial M}, \quad (5.7)$$

where $\text{vol}_{\partial M}$ denotes the canonical volume form of ∂M . Then we obtain

Theorem 5.1. *Let M be a three dimensional compact spin manifold with the boundary ∂M and the metric g^M as in Section 2, and \tilde{D} be the nonminimal de Rham-Hodge operators on \widehat{M} , then*

$$\widetilde{\text{Wres}}[\pi^+ \tilde{D}^{-1} \circ \pi^+(\tilde{D}^*)^{-1}] = (-\frac{1}{2} + \frac{i}{2})(\frac{1}{a^2} + \frac{1}{b^2})\pi\Omega_2\text{vol}_{\partial M}, \quad (5.8)$$

where $\text{vol}_{\partial M}$ denotes the canonical volume of ∂M .

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